Maximal ideals and discontinuous characters

By COLIN C. GRAHAM in Evanston (Ill., USA)

Let $\mu_1, \mu_2, \ldots$ be regular Borel measures on the non-discrete LCA group G, and let χ be a maximal ideal of the measure algebra $M(G)$, which is not in the dual group. Then there exists a maximal ideal χ' of $M(G)$ such that $\chi'(\mu_j)=\chi(\mu_j)$ $(1 \leqq j < \infty)$, and such that the restriction of χ' to $M_d(G)$ cannot be represented by a continuous character.

0. Introduction

Let Δ denote the maximal ideal space of the Banach algebra, $M(G)$, of regular Borel measures on a LCA group G, and $\hat{G}$ denote the dual group of G. Each $\chi \in \Delta$ gives rise (by restriction) to a maximal ideal χ^d of the discrete measures $M_d(G)$. Of course, $M_d(G)$ is $L^1(G_d)$ — L^1 of G with the discrete topology — so χ^d is represented on $M_d(G)$ by a character of G. Sometimes these characters are discontinuous. We prove:

Theorem. *Let G be a non-discrete LCA group, $\mu_1, \mu_2, \ldots$ a sequence in $M(G)$, and $\chi \in \Delta \setminus \hat{G}$. Then there exists $\chi' \in \Delta \setminus \hat{G}$ such that*

(i) $(\chi')^d$ *is discontinuous on G, and*

(ii) *if ν is in the L-algebra generated by $\mu_1, \mu_2, \ldots$, then $\chi'(\nu)=\chi(\nu)$.*

This affirms (in a strong way) a conjecture of GOLDBERG and SIMON [GS, p. 161]: $\{\chi \in \Delta : \chi^d$ is discontinuous$\}$ is dense in $\Delta \setminus \hat{G}$. This implies (as Goldberg and Simon point out): if $\mu \in M(G)$, then $\chi(\mu)=0$ for all $\chi \in \Delta \setminus G$ if and only if

$$\limsup_{x \to 0\ \chi \in \Delta} |\chi(\delta_x * \mu) - \chi(\mu)| = 0.$$

BROWN and MORAN [BM2] have shown that $\chi(\mu)=0$ for all $\chi \in \Delta \setminus \hat{G}$ if and only if $\lim_{x \to 0} (\chi(\delta_x * \mu) - \chi(\mu)) = 0$ for all $\chi \in \Delta \setminus \hat{G}$. It is their methods which we extend to obtain our theorem. We are grateful to Dr. G. Brown and Dr. W. Moran for several interesting conversations and letters.

In Section 1 we give notation and some preliminary lemmas. In Section 2 we prove the theorem.

While we have tried to make the paper self-contained, there may be points where the reader will wish to know more about L-algebras and generalized characters [S]. A good thorough introduction can be found in the report [T] of J. L. TAYLOR (which has also a good bibliography of the subject) and brief introduction in the introduction to the paper [BM 1].

1. Preliminaries

We shall write $\mu \perp \nu$ whenever $\mu, \nu \in M(G)$ are mutually singular, and $\mu \not\perp \nu$ otherwise. If μ is absolutely continuous with respect to ν, we write $\mu \ll \nu$. If $\mu \ll \nu$ and $\nu \ll \mu$, we write $\mu \approx \nu$. The unit point mass at $x \in G$ is denoted δ_x.

An *L-subalgebra* of $M(G)$ is a closed subalgebra A of $M(G)$ such that if $\mu \in A$ and $\nu \ll \mu$ then $\nu \in A$. Any maximal ideal χ of a L-subalgebra A of $M(G)$ restricts for each $\mu \in A$, to a linear functional χ_μ on $L^1(\mu) = \{\nu \in A \colon \nu \ll \mu\}$. Of course, χ_μ is given by integration against an element (denoted ambiguously by) χ_μ of $L^\infty(\mu)$.

Functions $\chi_\mu(t) \colon A \times G \to \mathbf{C}$ representing maximal ideals of A are called *generalized characters* [S]; they are characterized by these properties: for all $\nu, \mu \in A$, $x \in G$

$$|\chi_\mu(x)| \leqq 1 \quad \text{a.e. } d\mu \text{ for all } \quad \mu \in A, \quad x \in G;$$

$$\nu \ll \mu \quad \text{implies} \quad \chi_\nu = \chi_\mu \quad \text{a.e. } d\nu$$

$$\chi_{\mu * \nu}(x+y) = \chi_\mu(x)\chi_\nu(y) \quad \text{a.e.} \quad d\mu \times \nu.$$

It is easy to see that if χ_μ and χ'_μ are generalized characters, then the formulas $|\chi|_\mu(x) = |\chi_\mu(x)|$ and $(\chi\chi')_\mu(x) = \chi_\mu(x)\chi'_\mu(x)$ define generalized characters. Also, if $x_j \in (0, 1)$ and $\lim x_j = 0$, then $|\chi_\mu|^0 = \lim |\chi_\mu|^{x_j}$ exists and is, also, a generalized character which is one where $\chi_\mu \neq 0$ and zero where $\chi_\mu = 0$ (a.e. $d\mu$ for all μ).

A key step in the proof of the theorem is the following:

Lemma 1 ([BM 1, Theorem 1.2]). *Let χ be a maximal ideal of an L-subalgebra A of $M(G)$ such that $|\chi_\mu| = 1$ a.e. $d\mu$ for all $\mu \in A$. Then χ extends to a maximal ideal of $M(G)$.*

We need the two following technical lemmas. The first seems to be due to RAIKOV [Ra] and has been used recently in [D], [G, 3.1] and [BM 2].

Lemma 2. *Let $\mu \in M(G)$ be a singular measure (with respect to Haar measure). Then $\{x \colon \delta_x * \mu \not\perp \mu\}$ is a Borel subset of zero Haar measure.*

Proof. We may assume $\mu \geqq 0$ and $\|\mu\|=1$. Then $\delta_x * \mu \perp \mu$ iff $\|\mu-\delta_x*\mu\|=2$. Since f: $x \to \|\mu-\delta_x*\mu\|$ is the supremum of continuous functions, f is semicontinuous, so $\{x: f(x)<2\}=X$ is Borel.

Suppose X had non-zero Haar measure. (By considering any σ-compact open subgroup H of G which supports μ, we see that we may assume X has σ-finite Haar measure.) Then there exists a compact subset $A \subseteqq X$ of positive Haar measure α and such that

$$\sup\{\|\mu-\delta_x*\mu\| : x\in X\} \leqq 2-\varepsilon$$

for some $\varepsilon>0$. Let f be $1/\alpha$ times the characteristic function of A. Then it is easy to see that if g is continuous on G and vanishes at infinity, with $\|g\|_\infty \leqq 1$, then

$$\int g\,d(f*\mu-\mu)(y) = \int g(y)\int f(y-x)\,d\mu(x)\,dy - \int g(y)\,d\mu(y) =$$
$$= \int f(z)\left[\int g(x+z)\,d\mu(x) - \int g(x)\,d\mu(x)\right]dz \leqq$$
$$\leqq \int f(z)\left|\int g\,d(\delta_z*\mu-\mu)\right|dz \leqq (2-\varepsilon)\int f(z)\,dz = 2-\varepsilon.$$

The last equality follows from the translation-invariance of Haar measure and the fact that $\int f dz=1$. The last inequality follows from the choice of A. By taking a supremum over g, $\|g\|_\infty \leqq 1$, we see that $\|f*\mu-\mu\|<2$ which implies that μ is *not* singular ($f*\mu$ is absolutely continuous and $f*\mu$, μ are probability measures). Q.E.D.

Lemma 3. *Let H be a Borel subgroup of G with zero Haar measure. Then there exists a character of G/H which (when composed with the natural homomorphism $G \to G/H$) is not continuous.*

Proof. If H is closed, then G/H is a non-discrete LCA group and has a discontinuous character. It is easy to see that the resulting composition is not continuous.

If H is not closed, then there must be some character on the (for this purpose, discrete) group G/H whose composition with the projection of G on G/H is not continuous, for otherwise H is the intersection of the kernels of continuous characters, and therefore closed. Q.E.D.

2. Proof of theorem

Let $\omega_1=\sum_1^\infty (2^n\|\mu_n\|)^{-1}|\mu_n|$, and let ω_2 be the measure given by $d\omega_2=|\chi_{\omega_1}|^0 d\omega_1$ (since $\chi_{\omega_1}\in L^\infty(\omega_1)$, this makes sense). Now note that since $\chi \notin \hat{G}$, we see that if $\omega=\exp(\omega_2)$, then $\chi_\omega \neq 0$ a.e. $d\omega$. Therefore, ω is singular with respect to Haar measure. Also, $\omega^2 \approx \omega$.

We now apply Lemma 2: using the claim that $H=\{x\colon \delta_x*\omega \not\perp \omega\}$ is a subgroup of G. Indeed, it is obvious that $x\in H$ implies $-x\in H$:

$$\delta_x*\omega \not\perp \omega \quad \text{iff} \quad \omega = \delta_{-x}*\delta_x*\omega \not\perp \delta_{-x}*\omega.$$

If $x, y\in H$ then

$$\delta_{x+y}*\omega \approx \delta_x*\delta_y*\omega^2 = \delta_x*\omega*\delta_y*\omega \not\perp \omega^2 \approx \omega.$$

We let A be the L-subalgebra of $M(G)$ generated by ω and its translates $\delta_x*\omega$ as x runs through all of G. Note that every element of A is a sum

$$\sum_1^\infty \delta_{x(j)}*\nu_j, \quad \text{where } x(j)\in G, \text{ and } \nu \ll \omega.$$

Let γ be any character on G/H which is not continuous on G (when composed $G\to G/H$). We claim the map $\sum \delta_{x(j)}*\nu_j \to \sum (x(j), \gamma)\hat{\nu}_j(0)$ is a maximal ideal (generalized character) ϱ of A which satisfies the hypotheses of Lemma 1 and which has $\varrho_\omega=1$ a.e. $d\omega$ and has $|\varrho_\nu|=1$ a.e. $d\nu$ for all $\nu\in A$.

Before proving this claim, note that if it is proved, then Lemma 1 gives an extension ϱ' of ϱ to all of $M(G)$. Note that the extension has $\varrho'(\delta_x)=\varrho'(\delta_x*\omega)/(\varrho'(\omega))= =(x, \gamma)$ which is not continuous. We also claim that any extension ϱ' of ϱ to $M(G)$ has $\varrho'_\omega=1$ a.e. $d\omega$ so that if χ' is the product maximal ideal $\chi'_\nu=(\varrho'\chi)_\nu=\varrho'_\nu\chi_\nu$, then χ' and χ agree on the L-subalgebra of $M(G)$ generated by the measures $\mu_1, \mu_2, \dots$. These are enough to verify the theorem.

We now verify the first claim: that

$$\varrho\Big(\sum_1^\infty \delta_{x(j)}*\nu_j\Big) = \sum_1^\infty (x(j), \gamma)\,\hat{\nu}_j(0) \tag{1}$$

is well defined and multiplicative on A, and that

$$|\varrho_\nu| = 1$$

a.e. $d\nu$ for all $\nu\in A$. The last part is, of course, obvious. For the first, note that if $\delta_x*\nu_1 \not\perp \delta_y*\nu_2$, $(\nu_j\ll\omega)$ then $\delta_x*\omega \not\perp \delta_y*\omega$, so $x-y\in H$ and $(x-y, \gamma)=1$, that is $\delta_x*\nu_1 \not\perp \delta_y*\nu_2$ implies

$$\varrho(\delta_x*\nu_1+\delta_y*\nu_2) = (x, \gamma)\big(\hat{\nu}_1(0)+\hat{\nu}_2(0)\big).$$

Let $\nu\in A$ have two representations

$$\nu = \sum \delta_{x(j)}*\nu_j = \sum \delta_{y(j)}*\nu'_j. \tag{2}$$

We say ν_i and ν_k are *connected* if there exists a finite set $j(1), \dots, j(n)$ such that $j(1)=i$, $j(n)=k$ and

$$\delta_{x(j(s))}*\nu_{j(s)} \not\perp \delta_{x(j(s+1))}*\nu_{j(s+1)} \quad (1 \leqq s \leqq n-1). \tag{3}$$

By the least paragraph, if ν_i and ν_k are connected, then $(x(i), \gamma)=(x(k), \gamma)$. We rewrite the expressions for ν as

$$\nu = \sum_A \sum_{i\in A} \delta_{x(i)} * \nu_i = \sum_B \sum_{j\in B} \delta_{y(j)} * \nu'_j, \tag{4}$$

where each set $\{\nu_i : i\in A\}$ is a maximal connected subset of $\{\nu_i : 1\leqq i<\infty\}$, and each $\{\nu'_j : j\in B\}$ is maximal connected. Note that for each pair of sets

$$A_1 \neq A_2, \quad i\in A_1, \quad k\in A_2 \quad \text{imply} \quad \delta_{x(i)} * \nu_i \perp \delta_{x(k)} * \nu_k.$$

Thus, for each sum $\sum_{i\in A} \delta_{x(i)} * \nu_i$ there exists among the sets B a unique one $B=B_A$ such that

$$\sum_{i\in A} \delta_{x(i)} * \nu_i = \sum_{j\in B} \delta_{y(j)} * \nu'_j. \tag{5}$$

Of course, for some $i_0\in A$ and $j_0\in B$, $\delta_{x(i_0)} * \nu_{i_0} \not\perp \delta_{y(k_0)} * \nu'_{y_0}$ so $(x(i_0), \gamma)=(y(j_0), \gamma)$. By the choice of A and B,

$$\varrho\Big(\sum_{i\in A} \delta_{x(i)} * \nu_i\Big) = (x(\nu_j), \gamma) \sum \hat{\nu}_i(0) = (y(j_\nu), \gamma) \sum \hat{\nu}_j(0).$$

It is obvious from (5) that $\sum_{i\in A} \hat{\nu}_i(0) = \sum_{j\in B} \hat{\nu}'_j(0)$ (evaluate the Fourier—Stieltjes transform at the identity) so

$$\varrho\Big(\sum_{i\in A} \delta_{x(i)} * \nu_i\Big) = \varrho\Big(\sum_{g\in B} \delta_{y(j)} * \nu'_j\Big). \tag{6}$$

Since the correspondence between the sets A and B is one-to-one, f is well-defined. That ϱ is linear is obvious. That ϱ is multiplicative is the obvious computation from (1).

Finally, if ϱ' is any extension of ϱ to $M(G)$, then $\varrho_\omega=\varrho'_\omega=1$ a.e. $d\omega$.

We verify the last claim. Suppose ν is in the L-subalgebra of $M(G)$ generated by $\mu_1, \mu_2, \dots$. Then $\nu\ll\omega_1$, where ω is the measure defined at the beginning of this paragraph. Thus, it is enough to show $\chi'_{\omega_1}=\chi_{\omega_1}$ a.e. $d\omega_1$. Since $\chi'_{\omega_1}=\varrho'_{\omega_1}\chi_{\omega_1}$, it is enough to show $\{x: \chi_{\omega_1}(x)=0\}\cup\{x: \varrho'_{\omega_1}=1\}$ has ω_1-measure equal to $\|\omega_1\|$. But ω_2 is the restriction of ω_1 to $\{x: \chi_{\omega_1}(x)\neq 0\}$ and $\varrho'_{\omega_2}=\varrho'_\omega=\varrho_\omega=1$ a.e. $d\omega_2$ so $\varrho'_{\omega_1}=1$ where $\chi'_{\omega_1}\neq 0$. Thus $(\varrho'\chi)_{\omega_1}=\chi_{\omega_1}$ a.e. $d\omega_1$.

This completes the proof.

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NORTHWESTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS
EVANSTON, ILLINOIS 60201

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